Quantum Chaos and the Black Body Radiation

Giulio Casati

International Center for the Study of dynamical Systems, Università degli Studi dell' Insubria, Via Valleggio,11, 22100 Como, Italy
Istituto Nazionale di Fisica della Materia, Unità di Milano, Via Celoria 16, 20133 Milano, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy

(February 1, 2008)

We discuss a mechanical model which mimics the main features of the radiation matter interaction in the black body problem. The pure classical dynamical evolution, with a simple discretization of the action variables, leads to the Stefan- Boltzmann law and to the Planck distribution without any additional statistical assumption.

The problem of black body radiation occupies a central role in the history of physics. The long debate around its properties stimulated a profound revision of old and firmly established concepts and eventually led to the birth of quantum mechanics. Yet it is somehow unfortunate that such a profound revision of a fundamental theory has been based on a problem of so great complexity. Indeed, from the dynamical viewpoint, one is dealing with a nonlinear system with infinite degrees of freedom. Needless to say, such an infinite set of nonlinear differential equations is not solvable. What is more important, and perhaps less noticed, is that even the statistical description do not rest on solid grounds. As a matter of facts, classical ergodic theory which is now quite well developed, is valid for systems with a *finite* number N of degrees of freedoms. A main difficulty stems from the fact that the two limits $N \to \infty$ and $t \to \infty$ do not commute. The quantum statistical description is even more complex since it involves the additional difficulty that also the two limits $t \to \infty$, $\hbar \to 0$ do not commute.

In view of the relevance of the black body problem, it is perhaps worthwhile, after hundred years, a re-examination of the statistical properties with the help of modern computers and at the light of the recent progress in the study of nonlinear classical and quantum dynamical systems. This implies to tackle the problem of the statistical behaviour of infinite systems.

A distinctive feature of the radiation matter interaction in the black body is that each normal mode, or field's oscillator, interacts with the matter's degrees of freedom only, and that the strength of this interaction decreases as the mode's frequency increases. In this paper we introduce a simplified model which however shares the main features of the black body. We consider a system of N oscillators with mass $m_i = c/i^2$ and frequency $\omega_i = \sqrt{k/m_i} = \alpha i$ with $\alpha = \sqrt{k/c}$. When at its central position $x = 0$ each oscillator collides elastically with a particle of mass $M >> m_i$. Therefore the whole system is conservative with total energy

$$E = E_0 + \sum_{i=1}^{N} E_i = (1/2)MV^2 + \sum_{i=1}^{N} I_i\omega_i + h.c. \quad (1)$$

where E_0 is the energy of the heavy particle and I_i are the actions of the oscillators. The interaction between oscillators and the heavy particle is provided by the hardcore(h.c.) collisions. In our numerical computations we have taken the mass of the heavy particle $M = (\sqrt{5} + 1)/2$, $c = 0.51$, $k = 0.1$ so that $\omega_i = \alpha i$ with $\alpha \approx 0.443$. Notice that our system is of a billiard type and therefore the trajectory does not depend on the total energy E which is merely a scale factor. We recall that the one-dimensional system of two hard core point particles with fixed boundary conditions is equivalent to the billiard in a right triangle which is assumed to be ergodic and weakly mixing [1]. In our model the particle exchanges energy with each oscillator and moreover, after each collision, we assign at random, the sign of the velocity of the heavy particle. Our classical model has therefore a high degree of chaoticity. We would like to stress that for the purpose of the present paper we only need a mechanisms which conserves the total energy and allows energy exchange between the different oscillators. Therefore, we are not interested in the detailed motion of the heavy particle. Instead we need a sufficient degree of chaotic behaviour in order to ensure ergodicity of the entire system with $N + 1$ degrees of freedom [2].

The model system (1) is a mechanical version of the one dimensional black body problem discussed in [3] and one expects a similar statistical behaviour. Indeed we have numerically computed the time averaged energies of the oscillators and of the particle. As expected, we have observed that for any N, and independently of the initial condition, the system approaches the equipartition state in which the time averaged energies of the oscillators and of the particle are all equal (an example is given by the open squares in fig. 1) . Therefore, for any finite value of the total energy E, with increasing the number N of oscillators the temperature of the system $T = E/N$ will decrease down to zero as $N \to \infty$. The mechanisms through which such state is approached is the one already envisaged by Jeans [4]: energy flows from the matter(our particle) to higher and higher modes of the electromagnetic field(our oscillators) in such a way that the time averaged energy in each mode is zero while the total energy remains constant. Therefore the field continuously

absorbs energy from the matter, energy will move endless to higher and higher modes, and the whole system cools down to absolute zero temperature. This would be the behaviour according to classical laws. Fortunately however, this is not the case since our world is governed by quantum mechanics. The latter leads to the celebrated Planck distribution which, in one dimension, reads:

$$E_i = E(\omega_i) = \hbar\omega_i/(\exp\beta\hbar\omega_i - 1) \qquad (2)$$

The temperature $T = 1/\beta$ is finite and connected to the total energy E_f in the field (in the oscillators in our case) by the relation $E_f = \sigma T^2$ which is known as Stefan-Boltzmann law (in the one dimensional case).

Expression(2) has been derived by statistical methods. A *dynamical* derivation based on the quantum theory or on the numerical solution of the Schrodinger equation without any statistical assumption is still lacking. In this sense we are asking, for the quantum case, the same question that was posed about 50 years ago by Fermi, Pasta and Ulam [5] regarding the problem of classical equipartition (which is the limit of (2) for $\hbar \to 0$). Equipartition was a very well established consequence of classical statistical mechanics even though it rested on statistical assumptions. Fermi however, wanted to derive this property by a direct solution of Newton equations. The result was unexpected and Fermi considered it as one of the most important of his life. Indeed, as we now know, it is at the root of the modern field of nonlinear classical dynamics and chaos. It is therefore worthwhile to ask now a similar question: can we, by direct numerical integration of the time dependent Schrodinger equation, derive expression(2)?

A convenient guide in this direction can be provided by recent progress in the so-called field of quantum chaos. One of the main results in the study of this field is the quantum suppression of classically chaotic diffusion [6,7]. This phenomenon is known as quantum dynamical localization since is the dynamical analog of Anderson localization which takes place in disordered solids [8]. Localization represents a strong deviation from quantum ergodicity and only when localization length is larger than the sample size then eigenfunctions are extended, quantum ergodicity takes place and statistical methods can be applied. For example, in such situation, statistical properties of energy levels can be described by random matrix theory. Therefore, if the eigenfunctions are localized, then quantum evolution leads to a stationary state which is typically exponentially localized around the initially excited state and therefore strongly depends on the initial condition. On the other hand, if eigenfunctions are delocalised then the quantum stationary state is close to the corresponding classical one. In both cases it is not clear according to which quantum *dynamical* mechanisms Planck distribution sets in [9] .

In order to provide a clear answer to the above question it would be necessary to compute the quantum evolution

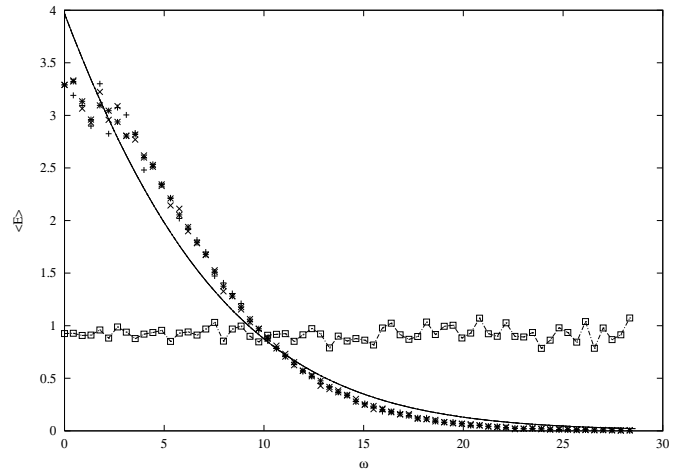for a system with many interacting particles like the system (1). This is a too difficult task.

FIG. 1. Time averaged energy of the particle E_0 and of the oscillators E_i, as a function of frequency ω for the discrete model with $N = 64$ and total energy $E = 60$. The averages are taken over a large number t of total collisions. Initial condition with almost all energy in the heavy particle: $t = 10^7$ (+), $t = 4(10^7)$ (*);classical model, $t = 4(10^7)$ (open squares). Initial condition with almost all energy in the oscillator i=32, $t = 10^7$(x). The full curve is the Planck law (2) with β given by eq(3) at $E = 60(\beta = 0.252)$.

In the following we present instead the results of a numerical integration of classical system (1) in which however we allow for the actions I_i to take only integers values. Namely, after each collision, which obeys the usual classical conservation laws, we substitute the values $I_j = E_j/\omega_j$ with the nearby integers n_j. The choice between the upper or the lower nearest integer is made at random after each collision. The roundoff energy of the oscillator is then given to the heavy particle in order to ensure total energy conservation. We will refer to this model as to the discrete model to distinguish it from the usual model(1) in which classical evolution is computed without any approximation. Our surmise, to be verified, is that such a simple discretization procedure will qualitatively reproduce the main results of an exact integration of Schrodinger equation (with $\hbar = 1$). Such possibility was also suggested in [10–12]. More recently we have found that application of such procedure to the standard map leads to exponential quantum localization [13]. Clearly the difference between quantum and classical mechanics goes much beyond the discrete nature of phase space. It is however our hope to gain a better understanding of their relationship.

In fig.1 we show the time averaged energies of the particle and of the oscillators, obtained with the above described numerical scheme for $N = 64$ and total energy $E = 60$. The thin curve is the theoretical Planck law (2)

2

with $\hbar = 1$ and β given by (3)with $E = 60$.

$$E = 1/\beta + \sum_{i=1}^{\infty} \omega_i/(\exp \beta \omega_i - 1) \qquad (3)$$

It is quite remarkable that, *independently on the initial condition*, the discrete model always reaches the same stationary distribution. We have also checked that the time-averaged values do not change by increasing the integration time. As an example we show in fig. 2 the time-averaged energies for few oscillators as a function of time measured in number of collisions, up to $t = 10^7$. These values do not change by further increasing the integration time.

We will not enter here in several questions of detail and simply notice that the overall agreement between the numerical results and the theoretical curve (2) (with the correct value of $\hbar = 1$)is sufficiently satisfactory. Solution of the classical model starting with the same initial conditions leads to equipartition as is shown by the open squares in fig.1.
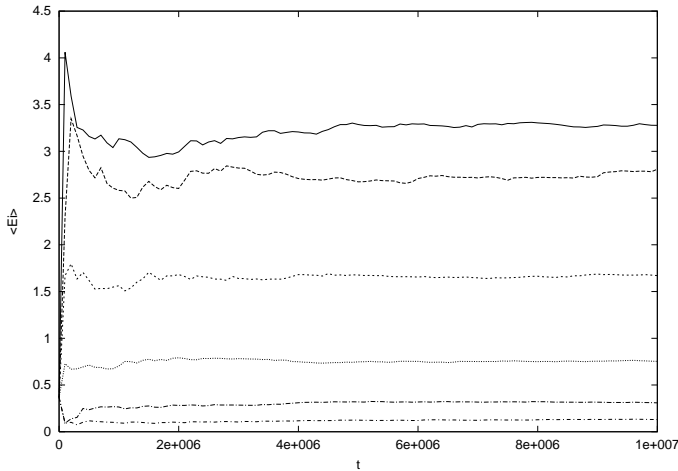

FIG. 2. Time-averaged energy $< E_i >$ of oscillators as a function of total number t of collisions. The curves refer to oscillators (starting from above): $i = 1$, $i = 8$, $i = 16$, $i = 24$, $i = 32$, $i = 40$.

In order to analyse the behaviour of the system as a function of the energy, at fixed N, we plot in fig.3 the total energy E of the system over the time averaged energy$< E_0 >$ of the particle. Clearly in our discret model the classical limit is reached for $E \to \infty$, since the effects of discretization become less and less important as the total energy is increased. Therefore, for sufficiently large energy, one expects equipartition which is given by the relation $< E >= (N+1) < E_0 >$. According to standard statistical mechanics, the temperature is defined as the average kinetic energy $T =< E_0) >$.
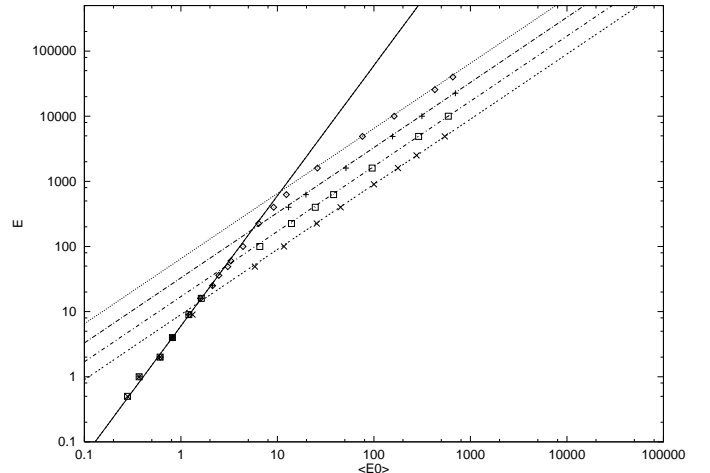

FIG. 3. The total energy E of the system over the average energy $< E_0 >= T$ of the particle for different N values. $N = 64$(open rombus), $N = 32(+)$, $N = 16$(open squares), $N = 8$(x). The solid line has slope two while the four parallel lines have slope one.

As we decrease energy, at fixed particle number N, we expect deviation from equipartition. It is really remarkable that integration of our discrete model leads to the Stefan-Bolzmann law which, in our one-dimensional case, implies that the total energy increases quadratically with the temperature. The solid line in fig. 3 is given by $E = \sigma< E_0 >^2 = \sigma T^2$ with $\sigma \approx 6$ (independent on N). The four parallel lines are given instead by $E = (N + 1) < E_0 > (N = 8, 16, 32, 64)$which is the exact equipartition law of classical mechanics. According to our data, the transition from Stefan-Boltzmann to equipartition is quite sharp and takes place at $E \approx N^2/\sigma$.
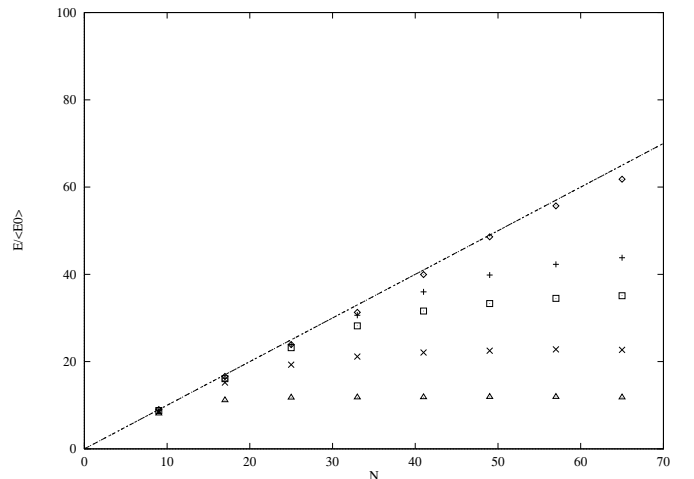

FIG. 4. Plot of $E/ < E_0 >$ versus N for different values of total energy E; $E = 25$(open triangles), $E = 100$(x), $E = 225$(open squares), $E = 400(+)$, $E = 1600$(open rombus). The dotted line has slope one and corresponds to equipartition.

In order to analyse instead the behaviour of the system as a function of the number N of oscillators, at fixed total energy E, we plot in fig.4 the quantity $E/<E_0>$ over N, for different values of the total energy E. The dotted line corresponds to equipartition and it is approached by decreasing N, at fixed energy E. Instead, by increasing N one approaches a constant value $\propto \sqrt{E}$. This is again in agreement with the Stefan-Boltzmann law. In summary, the results presented in figs 1-4 agree with the predictions of quantum mechanics. For a given N, at low temperature, the Stefan-Boltzmann law and the Planck distribution are obtained; as temperature is increased, one approaches the semiclassical limit in which equipartition takes place.

It may be interesting to examine this transition in relation to the general theory of quantum dynamical localisation. According to [14], the scaling *Ansatz* is equivalent to postulating the existence of a function $f(x)$ such that

$$l/N = f(x), x = \xi/N \tag{4}$$

where l is the actual localization length in the sample of finite size N, and ξ is the localization length for the infinite sample. As a measure of localization, following standard procedure, we take the inverse participation ratio:
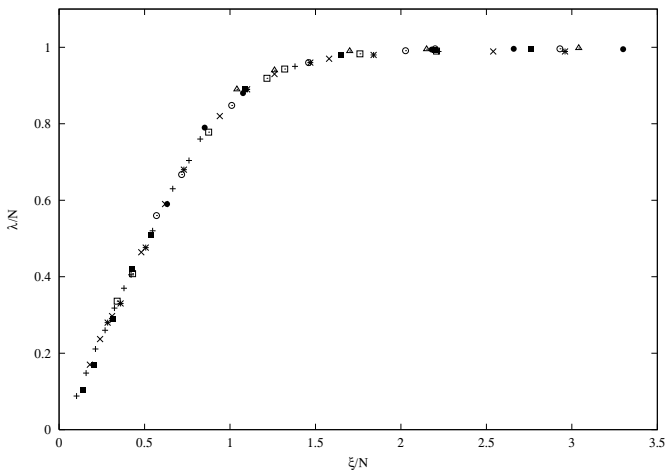
$$l = <E_f>^2 / \sum_{j=1}^{N} <E_j>^2 \tag{5}$$

FIG. 5. The actual localization length l/N for the discrete model over the localization length ξ/N for the infinite "sample" for different "sample sizes" N and for different energies E. $N = 64(+)$, $N = 56(x)$, $N = 48(*)$, $N = 40$(open squares), $N = 32$(full squares), $N = 24$(open circles), $N = 16$(full circles), $N = 8$(open triangles).

In fig.5 we plot the actual localization length l/N versus ξ/N where ξ is computed from the distribution (3).

The localisation length l is obtained from expression(5) where $<E_i>$ are the time-averaged energies of the oscillators computed for the discrete model and $<E_f>$ is the total average energy of the oscillators. A quite nice scaling behaviour is observed.

Certainly the results presented here require a better understanding. In particular it is not clear to what extent they are related to the peculiarity of the radiation matter interaction in which the field modes interact only through the matter. Needless to say the conclusions we have drawn here refer to the discrete model (1). It is our personal opinion that the solution of the time dependent Schrodinger equation will lead to the same qualitative results of figs. 1-5. This however needs a carefull study of true quantum mechanics of system(1) or of similar models.

[1] G. Casati and T. Prosen, Phys Rev Lett **83**(1999)4729.
[2] One may consider more realistic models than the one studied here. For example one may take the heavy particle moving between two fixed walls in one dimension or in a two-dimensional billiard. However the main results will remain unchanged. The main ingredients are the fact that the system is chaotic, the total energy is conserved and that the oscillators exchange energy only via interaction with the heavy particle. The choice of this particular model is only due to numerical convenience.
[3] G. Benenti, G. Casati and I. Guarneri, Europhys. Lett. **46** 307(1999).
[4] J.H.Jeans, *The dynamical Theory of Gases*, 2nd edition (Cambridge)1916.
[5] E. Fermi, J. Pasta, S. Ulam, and M. Tsingou, "Studies of Nonlinear Problems I", Los Alamos preprint LA-1940, 7 november 1955; See also E. Fermi, collected papers, vol.II (University of Chicago Press, Chicago, 1965), p. 978.
[6] G. Casati, B. V. Chirikov, J. Ford and F.M. Izrailev, in Lectures Notes in Physics, Vol 93 (Springer Verlag, Berlin, 1979), 334.
[7] G. Casati and B.V. Chirikov, in "Quantum Chaos", Cambridge University press, 1995.
[8] S. Fishman, D.R.Grempel, and R.E. Prange, Phys. Rev.Lett. **49**, 508(1982)
[9] M. Srednicki, Phys. Rev. E 50, 888 (1994);J.Phys.A29, 75 (1996);J.Phys. A 32, 1163 (1999).
[10] B.V. Chirikov, F.M.Izrailev and D.L. Shepelyansky, Sov. Sci. Rev. C2, 209 (1981)
[11] G.P.Berman,and A.R. Kolovsky, PhysicaD 17, 183(1985).
[12] G.P.Berman, A.R. Kolovsky, and F.M. Izrailev, Physica A 152 (1988)273
[13] G. Casati, unpublished
[14] J.L. Pichard, J. Phys. C **19**, 1519 (1986).